A SPATIAL LAMINAR BOUNDARY LAYER ON A STREAMLINE IN CONICAL EXTERNAL FLOW OF A UNIFORM GAS IN THE ABSENCE OF SOURCES AND SINKS

V. A. Bashkin

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 2, pp. 97-103, 1967

ABSTRACT: Theoretical study of a three-dimensional laminar boundary layer is a complex problem, but it can be substantially simplified in certain particular cases and even reduced to the solution of ordinary differential equations.

One such particular case is the flow of a compressible gas on a streamline in conical external flow. The case is of considerable practical importance because the local heat fluxes may take extremal values on such lines.

Such flow, except for the conical case, has been examined [1-4], and an approximate method has been given [1] on the basis of integral relationships and a special form for the approximating functions. A numerical solution has been given [2, 3] for such flow around an infinite cylinder. It was assumed in [1-3] that the Prandtl number and the specific heats were constant, and that the dynamic viscosity was proportional to temperature. Heat transfer has been examined [4] near a cylinder exposed to a flow of dissociated air.

Here we give results from numerical solution of a system of ordinary differential equations for the flow of a compressible gas in a laminar boundary layer on streamlines in conical external flow, with or without influx or withdrawal of a homogeneous gas. It is assumed that the gas is perfect and that the dynamic viscosity has a power-law temperature dependence.

§1. Struminskii [5] has derived the system of differential equations for a compressible gas in a three-dimensional laminar boundary layer, while Avduevskii [1] has given the system for the particular case of a boundary layer on a streamline in conical external flow.

The Lamé coefficients take the following form for a conical body in an (r, θ, z) coordinate system (in which r is distance along the surface of the body from the center of conicity, θ is the polar angle (triangular wing) or angle between the meridional planes (acute circular or elliptic cone), and z is perpendicular to the surface of the body:

$$h_1 = 1, \qquad h_2 = \psi(\theta)\ r, \qquad h_3 = 1, \tag{1.1}$$

in which $\psi(\theta)$ is a function whose form is determined by the geometry of the body.

We convert to Crocco variables and introduce dimensionless quantities as follows:

$$\tau_1 = \rho_e u_e^2 (1 + \alpha_1)^{0.5} R_r^{-0.5} Z(u_1), \qquad v = v_e v_1, \qquad u = u_e u_1,$$

$$H = H_e H_1, \qquad \rho = \rho_e \rho_1, \qquad \mu = \mu_e \mu_1, \qquad R_r = \rho_e u_e r / \mu_e,$$

$$\alpha_0 = \frac{u_e^2}{2H_e}, \qquad \alpha_1 = \frac{1}{u_e \psi^2} \frac{\partial v_e}{\partial \theta}\Big|_{\theta=0},$$

$$\alpha_2 = \frac{w_w \rho_w}{u_e \rho_e} \left(\frac{R_r}{1+\alpha_1}\right)^{0.5}, \tag{1.2}$$

and determine the functions $Z(u_1)$, $v_1(u_1)$, $H_1(u_1)$ on a streamline for conical external flow with transpiration of a homogeneous gas at the temperature of the body:

$$Z'' + \frac{1.5u_1 + \alpha_1 v_1}{1+\alpha_1} \frac{\rho_1 \mu_1}{Z} = 0,$$

$$v_1'' - \left\{ \frac{(u_1 + \alpha_1 v_1) v_1}{1 + \alpha_1} - \frac{1}{\rho_1} \right\} \frac{\rho_1 \mu_1}{Z^2} = 0,$$

$$H_1'' + (1 - P) H_1' Z'/Z = 2\alpha_0 (1 - P)(1 + u_1 Z'/Z), \tag{1.3}$$

the boundary conditions being

$$Z'(0) = \alpha_2, \quad v_1(0) = 0, \quad H_1(0) = H_{1w} = \text{const},$$
$$Z(1) = 0, \quad v_1(1) = H_1(1) = 1. \tag{1.4}$$

In (1.2)-(1.4), ρ is gas density, H is total enthalpy, μ is dynamic viscosity, u and v are the components of the velocity vector in the

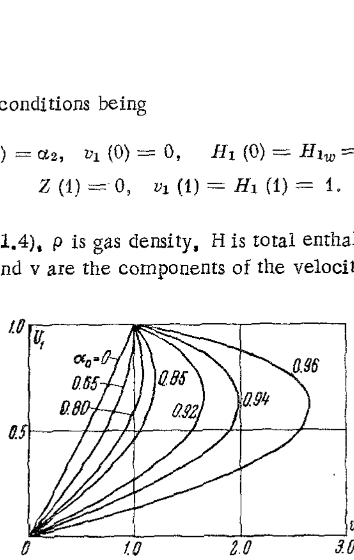

Fig. 1

longitudinal and transverse directions, τ_1 and τ_2 are the components of the frictional stress in those directions, P is the Prandtl number, R is the Reynolds number (as calculated from the flow parameters at the external edge of the boundary layer and the coordinate r), w_w is the speed of transpiration, subscript e characterizes the flow at the external edge of the boundary layer, the subscript w relates to the wall (surface of the body), and the subscript 1 denotes a dimensionless quantity referred to the value at the external edge.

If the solution to (1.3) is known, the local heat flux and stress components are

$$\tau_{1w} = 0.5\ \rho_e u_e^2 c_{f1} = \rho_e u_e^2 (1 + \alpha_1)^{0.5} R_r^{-0.5} Z(0),$$
$$\tau_{2w} = 0.5\ \rho_e u_e v_e c_{f2} = 0.5\ \rho_e u_e v_e\ c_{f1} v_1'(0),$$
$$q_w = \rho_e u_e H_e (1 + \alpha_1)^{0.5} R_r^{-0.5} Z(0) H_1'(0) P^{-1}. \tag{1.5}$$

System (1.3) with (1.4) has been integrated numerically by the Runge-Kutta method by means of second-order formulas, with a constant integration step $\Delta u_1 = 0.01$. The solution was derived by successive approximation, the boundary conditions at $u_1 = 1$ being obeyed to $\varepsilon = 10^{-5}$. The iteration was monitored via $Z(0)$, $v_1'(0)$, and $H_1'(0)$, the results being printed out when the difference between two successive approximations became less than $\varepsilon = 10^{-4}$.

The calculations were performed for the following ranges in the characteristic parameters: $0 \le \alpha_0 \le 0.96$ ($0 \le M_e \le 11$), $0 \le \alpha_1 \le \infty$, $-1 \le \alpha_2 \le +1$, and $0.05 \le H_{1w} \le 1$, with $\alpha_1 = 10^6$ being taken instead of ∞, which allowed the case to be calculated without altering the program.

It was assumed that the gas was perfect, that the Prandtl number was constant at 0.7, and that $\varkappa = 1.4$, with $\mu \propto T^{0.76}$.

§2. First we consider the case $\alpha_2 = 0$ (no transpiration).

We reduce $Z(u_1)$ to normal form by means of division by the value at the surface. Then $Z(u_1)/Z(0)$, $v_1(u_1)$, and $H_1(u_1)$ take values at the external and internal edges of the laminar boundary layer that are not dependent on the characteristic parameters.

It was found that α_1 had relatively little effect on $Z(u_1)/Z(0)$ and $H_1(u_1)$, while the deformation of these profiles in response to α_0 and H_{1w} was similar to that for planar and axially symmetric boundary layers at zero pressure gradient.

Figure 1 shows the profile of the secondary flow for various α_0 with $\alpha_1 = 0$ and $H_{1w} = 0.05$. For α_0 small (M for the flow small), the profile is monotonic, but changes occur as α_0 increases, and the speed of the secondary flow within the boundary layer exceeds that at the ex-

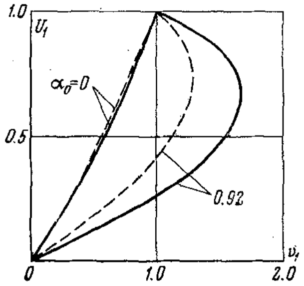

Fig. 2

ternal edge for $\alpha_0 \geq 0.7$. Increasing H_{1w} tends to accentuate the secondary flow, and this effect increases with α_0. These effects are physically expected, since the secondary flow in the boundary layer is due to unbalance between the pressure and the centrifugal forces in the external flow, and this unbalance increases with the speed of the external flow. Figure 2 shows the effects of α_1 on the profile of the secondary flow for $H_{1w} = 0.05$ and α_0 of 0 and 0.92 (the solid lines correspond to $\alpha_1 = 0$, while the dashed lines correspond to $\alpha_1 = \infty$).

Parameter α_1 scarcely influences the profile of the secondary flow for α_0 small, but the effects steadily increase with α_0. The secondary flow for a given α_0 is most prominent for $\alpha_1 = 0$. The reason for this response to α_1 is as follows. If $\alpha_1 = 0$, the spatial nature of the flow makes itself felt fully in the boundary layer, and the secondary flow is maximized for the given α_0. The flow in the boundary layer is degenerate for $\alpha_1 = \infty$, and so the secondary flow is less than for $\alpha_1 = 0$.

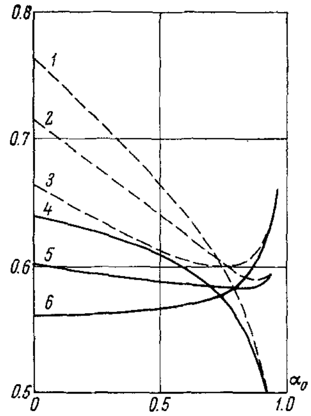

Fig. 3

This picture of the flow profile agrees well with with results from numerical integration for an infinite yawed cylinder [2, 3].

We need to know Z(0), H'_{1w}, and $v'_1(0)$ as functions of α_0, α_1, and H_{1w} in order to calculate the frictional resistance and heat-transfer coefficients.

Figure 3 shows Z(0) as a function of α_0 for given α_1 and for $H_{1w} = 0.05$ (the solid lines are for α_1 as follows: 4) 0, 5) 1, 6) ∞). The Z(0) for $\alpha_0 = 0$ decreases as α_1 increases in the presence of strong heat transfer, while Z(0) and α_1 increase together for $\alpha_0 = 0.96$.

This behavior of Z(0) arises because $\alpha_1 = 0$ corresponds to an acute circular cone at zero angle of attack, while $\alpha_1 = \infty$ corresponds to an infinite cylinder. Increase in α_0 in the first case (increase in the local M) causes reduction in the frictional resistance coefficient, and so an increase in α_0 causes a decrease in Z(0) for $\alpha_1 = 0$. The value of Z(0) near the planar critical point ($\alpha_0 = 0$) in the presence of strong heat transfer is close to the value for a planar plate and is less than that for a cone, where the flow is spatial. Then increase in α_1 for $\alpha_0 \approx 0$ causes a reduction in Z(0), while increase in α_0 for $\alpha_1 > 0$ accentuates the secondary flow, which increases Z(0) at first.

This increase ceases at a certain α_0, and for α_0 large enough Z(0) begins to increase with α_1. As the heat transfer is reduced (H_{1w} increased), the point of change in the dependence moves to the left, and Z(0) always increases with α_1 above a certain H_{1w}.

It is found that H'_{1w} is only very slightly dependent on α_1 (the deviation from the mean does not exceed ±0.5%), while the dependence on α_0 is very nearly linear.

Figure 4 shows that, for α_0 small, α_1 has very little effect on $v'_1(0)$, the effect becoming appreciable only for $\alpha_0 > 0.6$. The behavior for $\alpha_0 > 0.4$ has been described above in relation to the velocity profile. The deviation from the above relation for small α_0 arises because $v''_1(u_1) > 0$ near the external edge of the boundary layer, whereas this derivative is always negative for $\alpha_0 > 0.4$. This deviation arises from the effect of the Prandtl number P; the $v'_1(0)$ for $\alpha_0 = 0$ coincide for P = 1 and are not dependent on α_1, while $v'_1(0)$ always decreases as α_1 increases for P > 1.

We calculated q and the components of τ. Figure 3 shows $q_1 = q_w(\rho_e u_e H_e)^{-1} R_r^{0.5}(1 + \alpha_1)^{-0.5}$ as a function of α_0 for given α_1

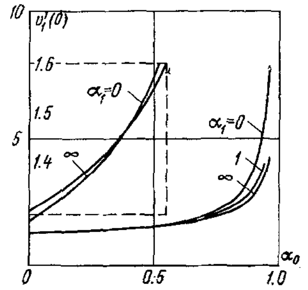

Fig. 4

with $H_{1w} = 0.05$ (the dashed lines are for α_1 as follows: 1) 0, 2) 1, 3) ∞). The behavior can be explained by reference to the limiting cases (α_1 equal to 0 and ∞).

§3. Transpiration in either sense has similar effects on the boundary layer, as Fig. 5 shows by reference to the secondary-flow profile as a function of α_2 for $H_{1w} = 0.05$, $\alpha_0 = 0.94$, and $\alpha_1 = 0$.

The effects on q and τ are similar; Fig. 6 shows q_1 as a function of α_0 for fixed values of α_1 and α_2 with $H_{1w} = 0.05$ (the solid line corresponds to $\alpha_1 = 0$, the dashed line to $\alpha_1 = 1$, and the dot-dash line to $\alpha_1 = \infty$). This shows that transpiration has effects analogous to those of a pressure gradient. These effects arise because the transpiration alters the thickness of the boundary layer and hence the velocity and temperature gradients, which affect τ and q.

Inward gas flow greatly reduces the local convective heat transfer, and so this provides an efficient method of protection for aircraft. The discussion of this topic falls into two parts. In the first stage, there is absorption of heat by the inward-flowing gas (rise from T_0 to T_w); in the second, the gas enters the boundary layer and reduces the local convective transfer.

These two processes may be considered via the equilibrium surface temperature of an acute circular cone of length l = 1 m (along the

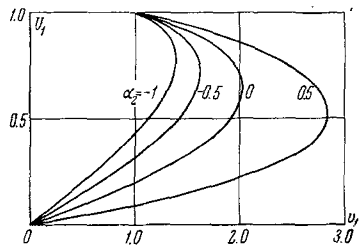

Fig. 5

surface) and semivertex angle $\vartheta = 30°$ moving supersonically ($M_\infty = 11$) at a height H = 25 km at zero angle of attack. We assume that the surface of the body is a perfect thermal conductor, the emissivity coefficient being $\varepsilon = 0.8$ and the initial gas temperature being $T_0 = 300°$ K. In the absence of inward transpiration, the surface temperature is governed by equality of the heat fluxes and is $T_{wp} = 2214°$ K. If we neglect heat absorption due to the heat capacity of the gas and take $\alpha_2 = 0.5$, outward transpiration of gas at the surface temperature reduces the latter to 1882° K. If we incorporate heating of the gas from $T_0 = 300°$ K to T_w, we get $T_{wp} = 1500°$ K. This requires a gas flow rate of only 0.68 kg/sec. This shows that transpiration provides efficient protection. However, a detailed study of this topic falls outside the scope of this paper.

These results apply only for a particular case, but they do provide some qualitative general conclusions as follows.

At a constant surface temperature and $\alpha_1 < \infty$, the gas flow rate varies as $\rho_w w_w = \alpha_2 u_e \rho_e (1 + \alpha_1)^{0.5} R_r^{-0.5}$, so most of the coolant enters the boundary layer on some initial section and takes up much of the heat flux, while the transpiration behind this plays a relatively minor part. In this case the surface temperature is constant throughout the transpiration region, while elsewhere it increases somewhat, though this rise is much less than in the absence of transpiration in the leading section.

For a given gas flow rate and given flow parameters at the external edge, this aftereffect of transpiration will be maximum for $\alpha_1 = 0$ and zero for $\alpha_1 = \infty$, since in the latter case the gas flow rate is everywhere the same, and all sections play the same part in the general balance. Physically, this is explained by the increase in removal of gas from the surface by the secondary flow as α_1 increases.

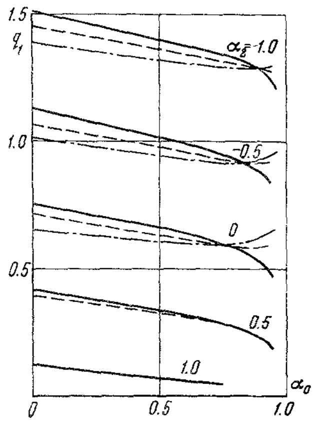

Fig. 6

Also, the aftereffect extends laterally for $\alpha_1 > 0$; if $\alpha_1 = \infty$, the aftereffect appears only in the direction of the secondary flow.

§4. We have seen that $\alpha_1 = 0$ corresponds to an acute circular cone at zero angle of attack, while $\alpha_1 = \infty$ corresponds to an infinitely long cylinder. These cases are of practical interest and will be examined in more detail, with particular attention to the heat transfer.

Figure 7 shows the local heat flux at the surface of a cone as the dependence of $c_{h\infty} R_{r\infty}^{0.5} = q_w R_{\infty r}^{0.5} [\rho_\infty u_\infty H_\infty (1 - H_{1w})]^{-1}$ on the angle ϑ for various M for rapid heat transfer ($H_{1w} = 0.05$, solid lines) and moderate heat transfer ($H_{1w} = 0.5$, dashed lines). The local transfer increases with ϑ, and the flux is maximum for the limiting ϑ at which the gas flow remains conical.

The following correlation formula applies for the local flux for cones with $\vartheta > 10°$ at hypersonic speeds with H_{1w} small:

$$\frac{q_w R_{\infty r}^{0.5}}{\rho_\infty u_\infty H_\infty} = 0.642\,(1 - H_{1w})\, M_\infty \sin\vartheta \sqrt{\cos\vartheta} \times$$
$$\times \left[H_{1w}\left(1 + \frac{\varkappa - 1}{2} M_\infty^2\right)\right]^{-0.12}. \qquad (4.1)$$

This formula is correct to ±8%. The following table gives the relative heat flux $k = q_{w\chi}/q_{w\chi=0}$ near the critical line on an infinite cylinder for $H_{1w} = 0.05$ and various M and χ.

$\chi°$	0	20	40	60	80	M_∞
k	1.0	0.919	0.705	0.0425	0.0166	3
k	1.0	0.913	0.686	0.0401	0.0130	5
k	1.0	0.908	0.672	0.0381	0.0115	15
k	1.0	0.908	0.671	0.0379	0.0114	25
$(\cos\chi)^{1.25}$	1.0	0.925	0.716	0.0420	0.0112	—

These values agree quite well with the results from $k = (\cos\chi)^{1.25}$, a formula that can be used near the critical line of a yawed cylinder.

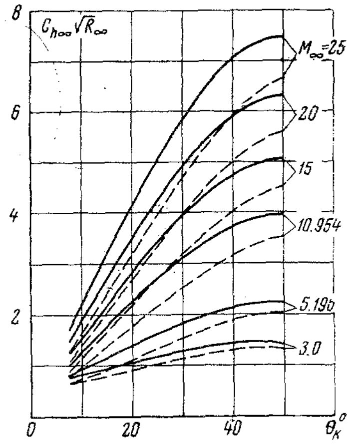

Fig. 7

(It has been suggested [2] that $k = \cos\chi$ should be used for this purpose.) It is clear that increase in χ greatly reduces the local heat flux, so the local heat flux at the leading rounded edge of a triangular wing may be reduced by increasing χ. However, it has been shown [6] by experiment for M of 4 to 10 for the critical line on a yawed cylinder that the laminar boundary layer becomes turbulent at a Reynolds number $R^* = \rho_e u_e \theta^*/\mu_e$ calculated from the flow parameters at the external edge and for a characteristic thickness

$$\theta^* = \int_0^\delta \rho_1 v_1 (1 - v_1)\, dz.$$

This transition begins at $R^* = 130$, and the flow is completely turbulent for $R^* \geq 450$, at which point the heat flux has increased by a factor of 4–5. For this reason, increase in χ reduces the heat flux only within certain limits under given conditions, and χ greater than a certain limit in fact increases the heat flux on account of turbulence in the boundary layer.

If we take the critical R^* as 130, we can determine the limiting χ as a function of the flight conditions. Figure 8 shows $R_{\infty D} = \rho_\infty u_\infty D/\mu_\infty$, in which D is the diameter of the cylinder, as a function of the limiting χ for various M with $H_{1w} = 0.05$ and $\alpha_2 = 0$.

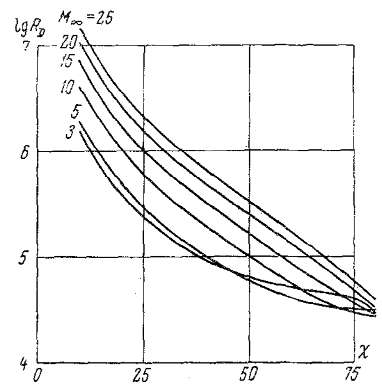

Fig. 8

If χ is near the limiting value, inward transpiration may actually increase the local heat flux on account of production of turbulence in the boundary layer. In that case, suctioning the gas may be advantageous, since it reduces θ^* and retards the onset of turbulence, which allows the limiting χ to be increased.

REFERENCES

1. V. S. Avduevskii, "Calculation of a three-dimensional laminar boundary layer on streamlines," Izv. AN SSSR, Mekhanika i Mashinostroenie, no. 1, 1962.

2. E. Reshotko and J. Beckwith, "Compressible laminar boundary layer over a yawed infinite cylinder with heat transfer and arbitrary Prandtl number," NACA Rep. 1379, 1958.

3. J. Beckwith, "Similar solution for the compressible boundary layer on a yawed cylinder with transpiration cooling," NASA TR R-42, 1959.

4. G. A. Tirskii, "Heat transfer near the leading edge of an infinitely long cylinder inclined to a flow of dissociated air," Izv. AN SSSR, Mekhanika i Mashinostroenie, no. 6, 1962.

5. V. V. Struminskii, "Equations for a three dimensional boundary layer in a compressible gas on an arbitrary surface," Dokl. AN SSSR, **114**, no. 2, 1957.

6. D. R. Topham and R. A. S. Grad, "A correlation of leading edge transition and heat transfer on swept cylinders in supersonic flow," J. Royal Aeronaut. Soc., no. 649, 1965.

20 July 1965 Moscow